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ON RATE-TYPE CONSTITUTIVE EQUATIONS AND THE ENERGY OF VISCOELASTIC AND VISCOPLASTIC MATERIALS

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Abstract—We discuss the qualitative behavior of the constitutive relation

$$\dot{\sigma} = E(\varepsilon, \sigma)\dot{\varepsilon} + G(\varepsilon, \sigma).$$

We show, for example, that this relation exhibits hypoelastic behavior under retardations of the time scale and rate-independent plastic behavior under accelerations of the time scale. We prove further that for a viscoelastic material governed by such a constitutive relation there exists a unique free energy. For a viscoplastic material, however, there are an uncountable infinity of free energies.

INTRODUCTION

In this paper we discuss materials defined by constitutive equations of the form,†

$$\dot{\sigma} = E(\varepsilon, \sigma)\dot{\varepsilon} + G(\varepsilon, \sigma) \tag{1}$$

relating the stress $\sigma(t)$ and the strain $\varepsilon(t)$. Relations of this type describe viscoelastic behavior when G is smooth, but when G is piecewise smooth with $G \equiv 0$ on a suitable region, (1) describes a rate-dependent theory of plasticity, which we refer to as viscoplasticity.

We here analyze the qualitative behavior of the relation (1). In particular, we show that under accelerations of the time scale (1) behaves like a hypoelastic material with constitutive equation

$$\dot{\sigma} = E(\varepsilon, \sigma)\dot{\varepsilon},$$

while under retardations (1) exhibits an approach to equilibrium and approximates rate independent plasticity theory.

We also discuss the construction of a free energy for a system governed by (1).‡ We prove that for a viscoelastic material a free energy exists and is unique (modulo a constant); for a viscoplastic material, however, there are an uncountably infinite number of free energies.

For convenience, we treat only the one-dimensional case, thereby avoiding the complications arising from the geometry of higher-dimensional theories (see Mihaïlescu-Suliciu and Suliciu [12]).

1. FUNDAMENTAL EQUATIONS. RESCALED PROCESSES

Consider the constitutive relation (1) with $E:\mathbf{R}^2 \to \mathbf{R}$ smooth§ and *strictly positive*, and $G:\mathbf{R}^2 \to \mathbf{R}$ locally Lipschitz.¶ We call E the instantaneous elasticity and G the relaxation modulus, a terminology motivated by our subsequent results.

For convenience, we restrict our attention to *strain paths* $\varepsilon:[0, \infty) \to \mathbf{R}$ which are smooth.

†See Sokolovskii [1, 2], Malvern [3, 4], Simmons, Hauser and Dorn [5], Lubliner [6], Cristescu [7, 8].

‡The problem of displaying a free-energy for hypoelasticity is well-discussed in the literature (see Ericksen [9], Bernstein and Ericksen [10], Coleman and Owen [11] and Mihaïlescu-Suliciu and Suliciu [12]).

§By "smooth" we always mean "of class C^1".

¶In what follows we allow the "states" (ε, σ) to assume arbitrary values in $\mathbf{R}^2$; our results are easily modified to include the case in which the states are restricted to lie in an open domain in $\mathbf{R}^2$.

Then given a strain path and an initial stress λ, (1) and the initial condition

$$\sigma(0) = \lambda$$

constitute an initial-value problem for the stress path σ. In view of our assumptions concerning E, G and ε, this problem has a unique solution $\sigma(t)$, at least within some non-empty time interval $0 \leq t < T$, which we will always suppose to be maximal. A pair (ε, σ) constructed in this manner will be called a *process* with *duration* T. If ε is constant then (ε, σ) is a *constant-strain process*.

We wish to consider processes obtained by retarding or accelerating a given strain path ε. Thus we rescale ε by defining $e_\alpha:[0,\infty)\to\mathbf{R}$, for $\alpha \in (0,\infty)$, by

$$e_\alpha(t) = \varepsilon(\alpha t).$$

Let s_α denote the stress path corresponding to e_α and a given initial stress λ, so that s_α solves the initial-value problem

$$\dot{s}_\alpha = E(e_\alpha, s_\alpha)\dot{e}_\alpha + G(e_\alpha, s_\alpha),$$

$$s_\alpha(0) = \lambda.$$

In order to compare the resulting stress paths s_α, we define

$$\sigma_\alpha(t) = s_\alpha(t/\alpha),$$

which returns the time scale to its original form. Then, clearly, σ_α is a maximal solution of

$$\dot{\sigma}_\alpha = E(\varepsilon, \sigma_\alpha)\dot{\varepsilon} + \frac{1}{\alpha}G(\varepsilon, \sigma_\alpha),$$

$$\sigma_\alpha(0) = \lambda. \tag{2}$$

We will also consider the instantaneous elastic solution σ_∞ which satisfies

$$\dot{\sigma}_\infty = E(\varepsilon, \sigma_\infty)\dot{\varepsilon},$$

$$\sigma_\infty(0) = \lambda.$$

We call $\{(\varepsilon, \sigma_\alpha)\} = \{(\varepsilon, \sigma_\alpha)|0 < \alpha \leq \infty\}$ defined in this manner a *family of rescaled processes*. An increase in α corresponds to an acceleration of the process. The next theorem shows that as $\alpha\to\infty$ the rescaled processes $(\varepsilon, \sigma_\alpha)$ tend monotonically to the instantaneous elastic "process" $(\varepsilon, \sigma_\infty)$.

Theorem 1. *Let* $\{(\varepsilon, \sigma_\alpha)\}$ *denote a family of rescaled processes, and suppose that for some time interval* $[0, T]$ *each of the rescaled processes lies in a region on which* $G \leq 0$ (resp. ≥ 0). *Then for each* $t \in [0, T]$, $\sigma_\alpha(t)$ *is a monotone increasing* (resp. *decreasing*) *function of* α *and*

$$\lim_{\alpha\to\infty} \sigma_\alpha(t) = \sigma_\infty(t).$$

Proof. We will consider only the case for which $G \leq 0$. If $\alpha > \beta$ and $\sigma_\alpha(t) = \sigma_\beta(t)$ at some time t, then, writing $E = E(\varepsilon(t), \sigma_\alpha(t))$ and $G = G(\varepsilon(t), \sigma_\alpha(t))$,

$$\dot{\sigma}_\alpha(t) = E\dot{\varepsilon}(t) + \frac{1}{\alpha}G \geq E\dot{\varepsilon}(t) + \frac{1}{\beta}G = \dot{\sigma}_\beta(t).$$

Thus $\dot{\sigma}_\alpha(t) \geq \dot{\sigma}_\beta(t)$ at any t for which $\sigma_\alpha(t) = \sigma_\beta(t)$; since $\sigma_\alpha(0) = \sigma_\beta(0)$, this implies $\sigma_\alpha(t) \geq$

$\sigma_\beta(t)$ on $[0, T]$. The same argument also shows that

$$\sigma_\infty(t) \geq \sigma_\beta(t)$$

on $[0, T]$.

To establish the approach to σ_∞ we fix $\alpha_0 > 0$ and confine our attention to $\alpha > \alpha_0$; this ensures that on $[0, T]$ all processes $(\varepsilon, \sigma_\alpha)$ lie within the rectangle $\mathscr{R}$ in $\mathbf{R}^2$ consisting of all (x, y) with

$$\min_{[0,T]} \varepsilon \leq x \leq \max_{[0,T]} \varepsilon, \qquad \min_{[0,T]} \sigma_{\alpha_0} \leq y \leq \max_{[0,T]} \sigma_\infty.$$

Let $K > 0$ and $M > 0$ denote upper bounds for $|\partial E/\partial\sigma|$ and $|G|$ on $\mathscr{R}$, and let $C/K > 0$ be an upper bound for $|\dot{\varepsilon}|$ on $[0, T]$. Then

$$h_\alpha = \sigma_\infty - \sigma_\alpha \geq 0$$

satisfies

$$\dot{h}_\alpha = \{E(\varepsilon, \sigma_\infty) - E(\varepsilon, \sigma_\alpha)\}\dot{\varepsilon} - \frac{1}{\alpha} G(\varepsilon, \sigma_\alpha) \leq C h_\alpha + \frac{1}{\alpha} M.$$

But $h_\alpha(0) = 0$; hence

$$h_\alpha(t) \leq \frac{M}{C\alpha}(e^{Ct} - 1),$$

and, since $h_\alpha \geq 0$, $h_\alpha(t) \to 0$ as $\alpha \to \infty$. □

An *instantaneous* (stress–strain) *curve* is a solution $\varepsilon \mapsto \sigma_I(\varepsilon)$—extended maximally forward and backward—of the differential equation

$$\frac{d\sigma_I(\varepsilon)}{d\varepsilon} = E(\varepsilon, \sigma_I(\varepsilon)).$$

These curves cover the stress–strain space $\mathbf{R}^2$, as every point $(\varepsilon_0, \sigma_0)$ has exactly one curve $\varepsilon \mapsto \sigma_I(\varepsilon) = \sigma_I(\varepsilon; \varepsilon_0, \sigma_0)$ passing through it. Moreover, each instantaneous elastic process

$$(\varepsilon, \sigma_\infty)$$

lies on an instantaneous curve; that is, for some σ_I,

$$\sigma_\infty(t) = \sigma_I(\varepsilon(t))$$

for all t.

2. APPROACH TO EQUILIBRIUM

We write

$$\text{Equil}$$

for the set of all *equilibrium points*; that is, the set of all $(\varepsilon, \sigma) \in \mathbf{R}^2$ for which

$$G(\varepsilon, \sigma) = 0.$$

An *equilibrium curve* is a C^2 function $\sigma_R: \mathbf{R} \to \mathbf{R}$ with the following properties:

(C_1) $(\varepsilon, \sigma_R(\varepsilon))$ is an equilibrium point for each $\varepsilon \in \mathbf{R}$;
(C_2) $\sigma_R'(\varepsilon) < E(\varepsilon, \sigma_R(\varepsilon))$ for each $\varepsilon \in \mathbf{R}$;
(C_3) σ_R crosses each instantaneous curve;
(C_4) $\liminf_{\varepsilon \to +\infty} \sigma_R(\varepsilon) > -\infty, \qquad \limsup_{\varepsilon \to -\infty} \sigma_R(\varepsilon) < +\infty.$

Condition (C_2) ensures that the slope of σ_R is less than that of any instantaneous curve which crosses it. Condition (C_3) is actually not needed until Section 4 (and until that section it sufficies to assume that σ_R be smooth rather than C^2). Finally, (C_4), while most reasonable, is technical and is needed only in the proofs of Theorems 2 and 3.

An equilibrium curve σ_R is *stable from above* (resp. *below*) if each constant-strain process (ε_0, σ) starting above (resp. below) σ_R has infinite duration and approaches σ_R in the sense that

$$\lim_{t\to\infty} \sigma(t) = \sigma_R(\varepsilon_0).$$

We now define the two materials of chief interest to us; the results of this section are most easily visualized in terms of these models. A *viscoelastic material* is one for which Equil consists of a single equilibrium curve which is stable from above and below. A *viscoplastic material* has two distinguished equilibrium curves σ_R^+ and $\sigma_R^-(\sigma_R^+ \geq \sigma_R^-)$ which bound Equil;† σ_R^+ is stable from above, σ_R^- from below (see Fig. 1).

To avoid repeated hypotheses we assume for the remainder of this section that σ_R is a given equilibrium curve. Since constant-strain processes satisfy

$$\dot{\sigma} = G(\varepsilon_0, \sigma),$$

an immediate consequence of our definition of stability is the following

Proposition. σ_R is stable from above (resp. *below*) *if and only if $G < 0$ above σ_R* (resp. *$G > 0$ below σ_R*).

A process (ε, σ_0) with σ_0 constant is called a *constant-stress process*; clearly it satisfies the differential equation

$$\dot{\varepsilon} = J(\varepsilon, \sigma_0),$$

where $J = -G/E$ (recall that $E > 0$).

While our definition of stability is expressed in terms of constant-strain processes (i.e. stress relaxation), it yields the stability of certain constant-stress processes (i.e. creep). Indeed, for σ_R stable from above, $J > 0$ above σ_R and all constant-stress processes starting above σ_R move in the direction of increasing strain. Thus, since $J = 0$ on σ_R, such processes will ultimately approach σ_R if (and only if) they begin in the set $\mathcal{R}^+$ of all $(\varepsilon_0, \sigma_0)$ above σ_R for which the ray $\{(\varepsilon_0 + \gamma, \sigma_0) | \gamma > 0\}$ intersects σ_R. A similar assertion applies to the set $\mathcal{R}^-$ of all $(\varepsilon_0, \sigma_0)$ below σ_R for which $\{(\varepsilon_0 - \gamma, \sigma_0) | \gamma > 0\}$ intersects σ_R. Thus we have the following

Proposition. Let σ_R be stable from above (resp. *below*). *Then every constant stress process (ε, σ_0) starting in $\mathcal{R}^+$* (resp. $\mathcal{R}^-$) *approaches σ_R as $t \to \infty$ in the sense that $\varepsilon(t)$ approaches a strain ε_0 with $\sigma_R(\varepsilon_0) = \sigma_0$.*

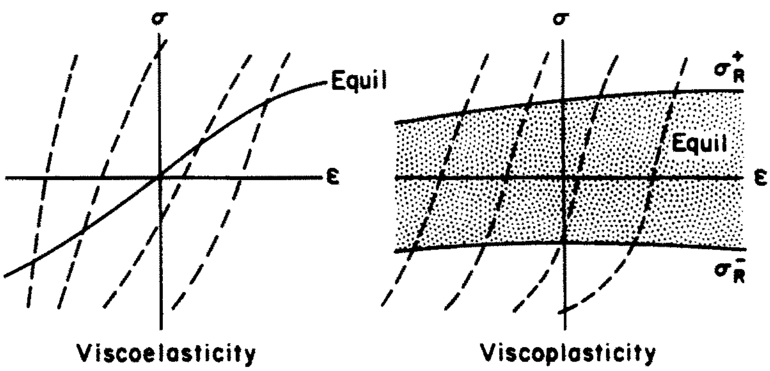

Fig. 1.

†What we call Equil is often referred to as the elastic range.

If σ_R is stable both constant strain and (appropriate) constant stress processes approach σ_R. Our next two theorems help to delineate the much larger class of processes attracted to σ_R. To state these results concisely we introduce the following notation.

For each $\delta > 0$, let

$$\mathcal{N}_\delta^+ = \{(\varepsilon, \sigma) \in \mathbf{R}^2 | \sigma \geq \sigma_R(\varepsilon) + \delta\}$$

(Fig. 2). We say that G is *uniformly negative above* σ_R if given any $\delta > 0$ there is a constant $K > 0$ such that

$$G(\varepsilon, \sigma) < -K$$

for all $(\varepsilon, \sigma) \in \mathcal{N}_\delta^+$. An analogous interpretation applies to the statement "G is *uniformly positive below* σ_R".

Theorem 2. *Assume that*: (a) *E is uniformly bounded*; (b) *G is uniformly negative above* σ_R (resp. *uniformly positive below* σ_R). *Let* (ε, σ) *be a process of infinite duration which starts above* (resp. *below*) σ_R *and has* (c) $\dot{\varepsilon} \geq 0$ (resp. ≤ 0) *and* $\lim_{t\to\infty} \dot{\varepsilon}(t) = 0$. *Then* (ε, σ) *approaches* σ_R *in the sense that*

$$\lim_{t\to\infty} |\sigma(t) - \sigma_R(\epsilon(t))| = 0.$$

Proof. We consider only the first case (for which G is uniformly negative above σ_R, etc.). Then by (c), (C_1) and (C_2), at any time t for which (ε, σ) contacts σ_R,

$$\dot{\sigma}(t) = E(\varepsilon(t), \sigma(t))\dot{\varepsilon}(t) > \sigma_R'(\varepsilon(t))\dot{\varepsilon}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma_R(\varepsilon(t));$$

hence (ε, σ) cannot cross σ_R. Thus it suffices to show that given any $\delta > 0$ there is a $T_0 > 0$ such that (ε, σ) lies outside of $\mathcal{N}_\delta^+$ for all $t > T_0$.

With this in mind, let

$$F(x, y, t) = E(x, y)\dot{\varepsilon}(t) + g(x, y), \tag{4}$$

so that the differential equation (1) becomes

$$\dot{\sigma}(t) = F(\varepsilon(t), \sigma(t), t). \tag{5}$$

Choose $\delta > 0$. Then there exist constants $T > 0$ and $M > 0$ such that

$$F(x, y, t) < -M \tag{6}$$

for all $(x, y) \in \mathcal{N}_\delta^+$ and $t > T$. Indeed, by hypotheses (a) and (b) in conjunction with (3) and (4) there exist constants $C > 0$ and $K > 0$ such that

$$F(x, y, t) < C\dot{\varepsilon}(t) - K$$

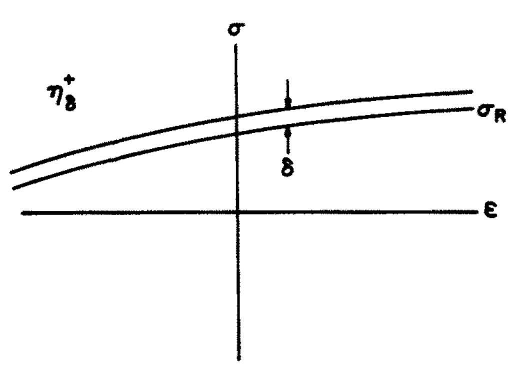

Fig. 2.

for all $(x, y) \in \mathcal{N}_\delta^+$, and, since $\dot{\varepsilon}(t) \to 0$ as $t \to \infty$, (6) follows. By (6), $F(x, y, t) < 0$ for all (x, y) on the boundary of $\mathcal{N}_\delta^+$ and all $t > T$. Thus (5) implies that if (ε, σ) leaves $\mathcal{N}_\delta^+$ at some $T_0 > T$, it will remain outside of $\mathcal{N}_\delta^+$ for all $t > T_0$. Thus the proof reduces to showing that (ε, σ) lies outside of $\mathcal{N}_\delta^+$ at some $T_0 > T$. Assume to the contrary that (ε, σ) lies within $\mathcal{N}_\delta^+$ for all $t > T$. Then by (5) and (6), $\dot{\sigma}(t) < -M$ for all $t > T$, so that $\sigma(t) \to -\infty$ as $t \to \infty$. But since (ε, σ) cannot cross σ_R, this contradicts (C_4). □

In the last section we discussed the behavior of rescaled processes $(\varepsilon, \sigma_\alpha)$ under an *acceleration* of the time scale. In particular, we showed that under appropriate assumptions $(\varepsilon, \sigma_\alpha)$ approaches the corresponding instantaneous curve as $\alpha \to \infty$. Our next theorem focuses on *retardations* of the time scale and asserts that under certain conditions $(\varepsilon, \sigma_\alpha)$ approaches the equilibrium curve σ_R as $\alpha \to 0$.

Theorem 3. *Assume that*: (a) *E is uniformly bounded*; (b) *G is uniformly negative above* σ_R (resp. *uniformly positive below* σ_R). *Let* $\{(\varepsilon, \sigma_\alpha)\}$ *be a family of rescaled processes which start above* (resp. *below*) σ_R, *which are of infinite duration for* α *sufficiently small, and which have* (c) $\dot{\varepsilon} \geq 0$ (resp. ≤ 0) *and uniformly bounded. Then given any* $\delta > 0$ *there exists a* $\beta > 0$ *and a time* $T > 0$ *such that all processes* $(\varepsilon, \sigma_\alpha)$ *with* $\alpha < \beta$ *satisfy*

$$|\sigma_\alpha(t) - \sigma_R(\varepsilon(t))| < \delta \tag{7}$$

for all $t > T$.

Proof. We consider only the first case. Our proof proceeds as before. Clearly, $(\varepsilon, \sigma_\alpha)$ cannot cross σ_R. Further, we define

$$F_\alpha(x, y, t) = E(x, y)\dot{\varepsilon}(t) + \frac{1}{\alpha}G(x, y),$$

so that (2) becomes

$$\dot{\sigma}_\alpha(t) = F_\alpha(\dot{\varepsilon}(t), \sigma_\alpha(t), t).$$

Choose $\delta > 0$. Then by (a), (b) and (c), there exist constants $\beta > 0$ and $M > 0$ such that

$$F_\alpha(x, y, t) < -M$$

for all $(x, y) \in \mathcal{N}_\delta^+$, $\alpha \leq \beta$ and $t > 0$. This result and arguments similar to those used in the proof of Theorem 2 imply that there exists a constant $\beta > 0$ (chosen small enough that all $(\varepsilon, \sigma_\alpha)$ with $\alpha \leq \beta$ have infinite duration) such that $(\varepsilon_\beta, \sigma)$ cannot remain in $\mathcal{N}_\delta^+$ for all time, and such that once $(\varepsilon, \sigma_\beta)$ leaves $\mathcal{N}_\delta^+$, it can never return. Thus there exists a $T > 0$ such that (7) holds for $\alpha = \beta$ and $t > T$. But by Theorem 1, $\sigma_\alpha \leq \sigma_\beta$ if $\alpha \leq \beta$; hence (7) holds for all $\alpha \leq \beta$ and $t > T$. □

The foregoing proof also yields the following

Corollary. Under the hypotheses of Theorem 3, *given any* $\delta > 0$, *if* (7) *is satisfied at* $t = 0$, *then* (7) *is satisfied for all t provided* α *is sufficiently small.*

Thus we can approximately traverse the equilibrium curve provided we proceed sufficiently slowly.

Remark 1. There is a *rate-independent theory* corresponding to the theory we discuss. For our viscoelastic model it describes a hypoelastic material with modulus E; for our viscoplastic model it reduces to an elasto–plastic (possibly work-hardening) material with elastic modulus E and yield "surface" $\sigma_R^\pm$. In either case the rate-independent theory produces processes which travel along instantaneous curves σ_I within Equil and on the equilibrium curves $\sigma_R^\pm$ at the boundary of Equil. Thus the above corollary states that in sufficiently slow processes the predictions of this theory approximate those of the corresponding rate-independent theory.

Remark 2. In many applications the assumption that E is uniformly bounded might be too strong. Consider the weaker hypothesis: (a′) E is uniformly bounded on a strip $S = \{(\varepsilon, \sigma) \in \mathbf{R}^2 \mid |\sigma - \sigma_R(\varepsilon)| < \delta\}$ for some $\delta > 0$. Then Theorems 2 and 3 remain valid provided the underlying processes are required to begin in S and have $\sup_{t \geq 0} |\dot{\varepsilon}(t)|$ sufficiently small.

Our results are most easily displayed in terms of the viscoplastic material discussed earlier. The dotted curves in Fig. 3 represent instantaneous curves. A process starting at A with $\dot{\varepsilon} > 0$ will proceed along an instantaneous curve until it reaches B, at which point it will diverge from this curve. A family of rescaled processes starting at C with $\dot{\varepsilon} > 0$ will appear as shown with C_1 representing a faster process than C_2, etc. As the processes are run faster and faster the curves will tend to the instantaneous curve (Theorem 1); on the other hand, as the processes are run slower the curves approach the equilibrium curve σ_R^+ (Theorem 3). Analogous assertions apply to the processes starting at A, D or E with $\dot{\varepsilon} < 0$.

We close this section with some remarks concerning the relaxation modulus G. In the literature G is often taken in the form

$$G(\varepsilon, \sigma) = -\kappa[\sigma - \sigma_R(\varepsilon)]$$

with $\kappa > 0$ a constant. Our next result helps to motivate this constitutive assumption, at least for behavior close to equilibrium.

Proposition. Assume that: (a) $G < 0$ *above* σ_R (resp. $G > 0$ *below* σ_R); (b) *G is smooth up to σ_R on the region above* (resp. *below*) σ_R. *Choose a strain ε_0 and let $\sigma_0 = \sigma_R(\varepsilon_0)$. Then there exists a constant $\kappa = \kappa(\varepsilon_0) \geq 0$ such that*

$$G(\varepsilon, \sigma) = -\kappa[\sigma - \sigma_R(\varepsilon)] + o(\delta)$$

as

$$\delta = |\varepsilon - \varepsilon_0| + |\sigma - \sigma_0|$$

approaches zero with (ε, σ) above (resp. *below*) σ_R. *If G is smooth in a neighborhood of $(\varepsilon_0, \sigma_0)$, then this estimate holds as $\delta \to 0$ with no restriction on (ε, σ).*

Proof. Simply expand

$$H(\varepsilon, \tau) = G(\varepsilon, \sigma_R(\varepsilon) + \tau)$$

in a Taylor series about $(\varepsilon, \tau) = (\varepsilon_0, 0)$ (on the appropriate halfspace) and use the fact that

$$H(\varepsilon, 0) = \frac{\partial}{\partial \varepsilon} H(\varepsilon, 0) = 0 \tag{8}$$

for all ε. The constant κ is $\partial H/\partial \tau$ evaluated at $(\varepsilon_0, 0)$; this derivative is ≥ 0 by (a). □

In terms of G, $(8)_2$ states that

$$\frac{\partial G}{\partial \varepsilon}(\varepsilon, \sigma_R(\varepsilon)) + \sigma_R'(\varepsilon)\frac{\partial G}{\partial \sigma}(\varepsilon, \sigma_R(\varepsilon)) = 0;$$

this relation is often assumed in the literature.

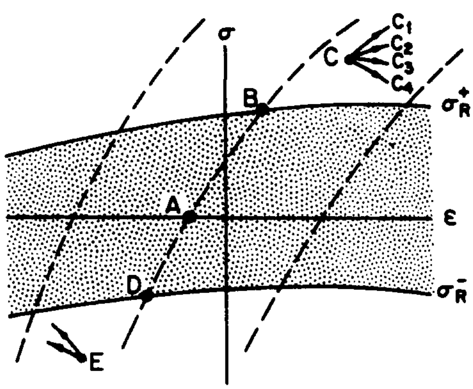

Fig. 3.

3. FREE ENERGIES

By a *free energy* we mean a smooth function $\psi:\mathbf{R}^2\to\mathbf{R}$ that satisfies

$$\psi(\varepsilon,\sigma)^{\cdot}\le\sigma\dot{\varepsilon} \tag{9}$$

on each process (ε,σ).

Proposition. A smooth function $\psi:\mathbf{R}^2\to\mathbf{R}$ *is a free energy if and only if*

$$\frac{\partial\psi}{\partial\varepsilon}+E\frac{\partial\psi}{\partial\sigma}=\sigma, \qquad G\frac{\partial\psi}{\partial\sigma}\le 0. \tag{10}†$$

Proof. By (1),

$$\psi(\varepsilon,\sigma)^{\cdot}=\frac{\partial\psi}{\partial\varepsilon}\dot{\varepsilon}+\frac{\partial\psi}{\partial\sigma}\dot{\sigma}=\left(\frac{\partial\psi}{\partial\varepsilon}+E\frac{\partial\psi}{\partial\sigma}\right)\dot{\varepsilon}+G\frac{\partial\psi}{\partial\sigma},$$

so that

$$\psi(\varepsilon,\sigma)^{\cdot}-\sigma\dot{\varepsilon}=\left(\frac{\partial\psi}{\partial\varepsilon}+E\frac{\partial\psi}{\partial\sigma}-\sigma\right)\dot{\varepsilon}+G\frac{\partial\psi}{\partial\sigma}, \tag{11}$$

and (10) implies (9). On the other hand, given any point (ε,σ) we can find a process with these values and $\dot{\varepsilon}$ arbitrary; hence (9) and (11) imply (10). □

Remark. To construct a free energy it is necessary to solve the partial differential equation $(10)_1$. Note that the characteristic curves of this equation are exactly the instantaneous stress-strain curves discussed in Section 1.

For the remainder of the section ψ is a free energy and σ_R is an equilibrium curve. For convenience, given any function $f(\varepsilon,\sigma)$, we write

$$f_R(\varepsilon)=f(\varepsilon,\sigma_R(\varepsilon))$$

for the values of f on σ_R. In particular, ψ_R represents the ***equilibrium free energy.***

Proposition. If σ_R *is stable from above* (resp. *below*), *then*

$$\psi_R(\varepsilon)\le\psi(\varepsilon,\sigma) \tag{12}‡$$

for all (ε,σ) *above* (resp. *below*) σ_R.

Proof. Let σ_R be stable from above. Then $G<0$ above σ_R (see the first proposition in Section 1) and $(10)_2$ yields $\partial\psi/\partial\sigma\ge 0$, which implies (12). □

If σ_R is stable from both sides, then $\psi(\varepsilon,\sigma)$ as a function of σ has a minimum on σ_R so that

$$\left(\frac{\partial\psi}{\partial\sigma}\right)_R=0. \tag{13}$$

Thus $(10)_1$ implies

$$\sigma_R=\left(\frac{\partial\psi}{\partial\varepsilon}\right)_R.$$

†See Nunziato and Drumheller[13], Theorem 1.

‡See Numziato and Drumheller[13], Theorem 3; Gurtin and Suliciu[14], Theorem 6.1.2.

But by the chain-rule,

$$\psi_R' = \left(\frac{\partial\psi}{\partial\varepsilon}\right)_R + \sigma_R'\left(\frac{\partial\psi}{\partial\sigma}\right)_R = \left(\frac{\partial\psi}{\partial\varepsilon}\right)_R,$$

and we have the following

Corollary.† *If σ_R is stable from both sides, then*

$$\sigma_R = \psi_R'.$$

This result is the classical thermostatic relation giving the equilibrium stress as the derivative of the equilibrium free energy. As our next result shows, a much weaker condition follows when σ_R is assumed stable from only one side.

Corollary.‡ *If σ_R is stable from above* (resp. *below*), *then*

$$\sigma_R \geq \psi_R' \qquad (\text{resp. } \sigma_R \leq \psi_R').$$

Proof. By $(10)_1$ and $(13)_1$,

$$\sigma_R = E_R\left(\frac{\partial\psi}{\partial\sigma}\right)_R + \left(\frac{\partial\psi}{\partial\varepsilon}\right)_R = \psi_R' + (E_R - \sigma_R')\left(\frac{\partial\psi}{\partial\sigma}\right)_R. \tag{14}$$

But by the last proposition, $(\partial\psi/\partial\sigma)_R \geq 0$ (resp. ≤ 0), and assumption (C_2) ensures that $E_R \geq \sigma_R'$, so the desired results follow. □

4. EXISTENCE OF FREE ENERGY

In this section we examine the question of existence and uniqueness of free energies; by "*unique*" we shall always mean "unique modulo an additive constant".

Recall that an equilibrium curve σ_R crosses each instantaneous curve, and $E_R > \sigma_R'$. In view of the remark made in the last section, this implies that all characteristic curves of $(10)_1$ cross σ_R, and that σ_R itself is nowhere characteristic.

We assume, throughout this section, that E is of class C^2.

Theorem 4. (Existence and uniqueness of free energy for a viscoelastic material.) *Assume that* Equil *is a single equilibrium curve σ_R, and that*

$$G < 0 \textit{ above } \sigma_R \textit{ and } G > 0 \textit{ below } \sigma_R. \tag{15}§$$

Then there exists a unique free energy.

Proof. If a free energy exists, clearly it must satisfy (13); therefore we seek a solution of $(10)_1$ subject to $\partial\psi/\partial\sigma = 0$ on σ_R. This problem has a unique solution ψ of class C_2 on $\mathbf{R}^2$, and we have only to show that ψ satisfies $(10)_2$. Fix $(\varepsilon_0, \sigma_R(\varepsilon_0))$ on σ_R, consider the characteristic curve $\varepsilon \mapsto \sigma_I(\varepsilon)$ passing through $(\varepsilon_0, \sigma_R)\varepsilon_0))$, and for any function $f(\varepsilon, \sigma)$ define

$$f_I(\varepsilon) = f(\varepsilon, \sigma_I(\varepsilon)).$$

Then

$$w = \left(\frac{\partial\psi}{\partial\sigma}\right)_I \tag{16}$$

satisfies

$$w' = \left(\frac{\partial^2\psi}{\partial\sigma\partial\varepsilon} + E\frac{\partial^2\psi}{\partial\sigma^2}\right)_I.$$

†See [13], Theorem 4.
‡See [14], Theorem 6.1.3.
§By the first proposition of Section 2, (15) can be replaced by the requirement that σ_R be stable from above and below.

But by $(10)_1$,

$$\frac{\partial^2\psi}{\partial\sigma\partial\varepsilon}+\frac{\partial E}{\partial\sigma}\frac{\partial\psi}{\partial\sigma}+E\frac{\partial^2\psi}{\partial\sigma^2}=1,$$

so that

$$w'+\left(\frac{\partial E}{\partial\sigma}\right)_I w=1.$$

Further, $w(\varepsilon_0)=0$, since $\partial\psi/\partial\sigma$ vanishes on σ_R; hence the foregoing differential equation has the solution

$$w(\varepsilon)=e^{-\varphi(\varepsilon)}\int_{\varepsilon_0}^{\varepsilon}e^{\varphi(\eta)}\,d\eta, \tag{17}$$

where

$$\varphi(\xi)=\int_{\varepsilon_0}^{\xi}\left(\frac{\partial E}{\partial\sigma}\right)_I(\eta)\,d\eta.$$

By (16) and (17), $\partial\psi/\partial\sigma>0$ above σ_R and $\partial\psi/\partial\sigma<0$ below σ_R. Thus by (15), $(10)_2$ is satisfied. □

Remark. It is clear from the above proof that *the free energy is independent of the relaxation modulus G*. Indeed, for given functions E and σ_R there exists exactly one free energy for the entire class of relaxation moduli G consistent with (15).

Theorem 5. (Existence of free energies for a visocoplastic material.) *Assume that the set* Equil *lies between two regular equilibrium curves* σ_R^+ *and* σ_R^- *with* $\sigma_R^+\geq\sigma_R^-$ *and* $\sigma_R^+\not\equiv\sigma_R^-$. *Assume further that*

$$G<0 \textit{ above } \sigma_R^+ \textit{ and } G>0 \textit{ below } \sigma_R^-. \tag{18}$$

Then there exist an uncountably infinite number of free energies.

Proof. Assume that (18) holds. Let $\psi^\pm$ be the solution of $(10)_1$ with $\partial\psi/\partial\sigma=0$ on $\sigma_R^\pm$. The calculation given in the proof of the last theorem then shows that both ψ^+ and ψ^- satisfy $(10)_2$, as does any convex combination

$$\psi=\lambda\psi^++(1-\lambda)\psi^-$$

with $\lambda\in(0,1)$. □

Remark. It is not difficult to show that, when E is constant, (15) (or 18) is also necessary for the existence of a free energy and that instability above and below ($G>0$ above σ_R^+ and $G<0$ below σ_R^-) ensures nonexistence.

Remark. It is clear from the above proof that for any free energy ψ, $\partial\psi/\partial\sigma$ cannot vanish on both σ_R^+ and σ_R^-. Thus (14) implies that σ_R^+ and σ_R^- cannot both be derivatives of the equilibrium free energy. Thus *for a viscoplastic material the classical thermostatic relation* $\sigma_R=\psi_R'$ *cannot hold at every equilibrium point.*

To illustrate the construction of the free energy, consider the Sokolovskii [3, 4] model of viscoplasticity, which is specified by a function G (consistent with (18), a constant instantaneous elasticity,

$$E(\varepsilon,\sigma)=E_0>0,$$

and an equilibrium set of the form

$$\text{Equil}=\{(\varepsilon,\sigma)\in\mathbf{R}^2|\quad|\sigma|\leq\sigma_Y\}.$$

Here $\sigma_Y > 0$ is a constant called the *yield stress*. Note that

$$\sigma_R^{\pm}(\varepsilon) = \pm\sigma_Y$$

for all ε. For such a material the free energies $\psi^{\pm}$, which have $\partial\psi^{\pm}/\partial\sigma = 0$ on $\sigma_R^{\pm}$, are given by

$$\psi^{\pm}(\varepsilon, \sigma) = \frac{1}{2E_0}\sigma^2 \pm \frac{1}{E_0}\sigma_Y(E_0\varepsilon - \sigma).$$

Of course, a viscoelastic case is included by setting $\sigma_Y = 0$.

Concerning Remark 1 (on p. 612) the rate independent theory corresponding to the above viscoplasticity is the theory of elastic perfectly-plastic materials. If the equilibrium set is

$$\text{Equil} = \{(\varepsilon, \sigma) \in \mathbf{R}^2 | \quad |\sigma - E_1\varepsilon| \le \sigma_Y\},$$

where $E_0 > E_1 = \text{const.} > 0$ and G is defined consistent with (18), then the viscoplastic model corresponds to the rate independent model describing linear work-hardening materials with idealized Bauschinger effect.

(The corresponding free energies are

$$\psi^{\pm}(\varepsilon, \sigma) = \frac{1}{2E_0}\sigma^2 + \frac{1}{E_0 - E_1}\left[\frac{E_1}{2E_0}(E_0\varepsilon - \sigma)^2 \pm \sigma_Y(E_0\varepsilon - \sigma)\right]$$

and

$$\sigma_R^{\pm}(\varepsilon) = E_1\varepsilon \pm \sigma_Y.)$$

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